Proving Goldbach's Weak Conjecture

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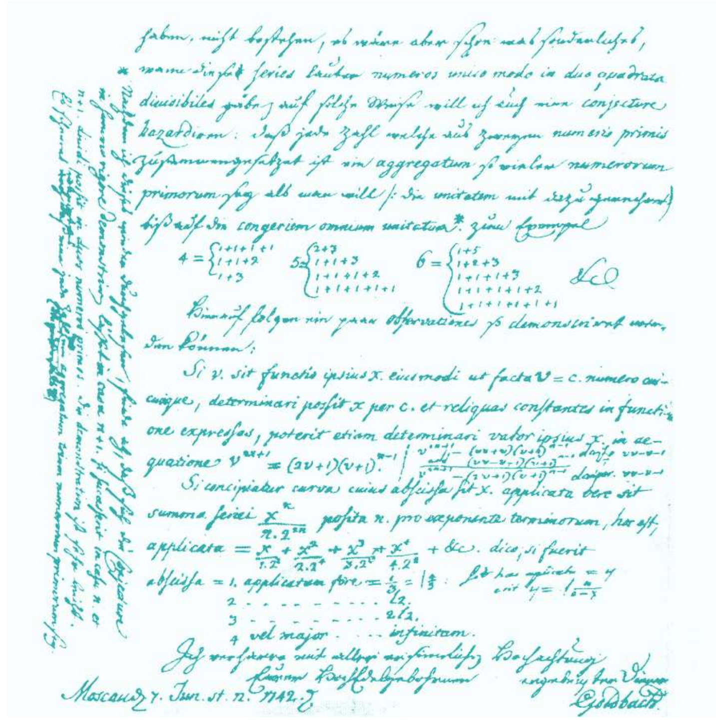
Introduction

In 1742, Christian Goldbach wrote to Leonhard Euler and observed that if he picked any odd number > 5, it seemed he could always write it as a sum of three primes in at least one way. For example

$$1,000,001 = 333,323 + 333,337 + 333,341.$$

His guess that this would always be the case became known as the weak (or ternary) Goldbach conjecture.

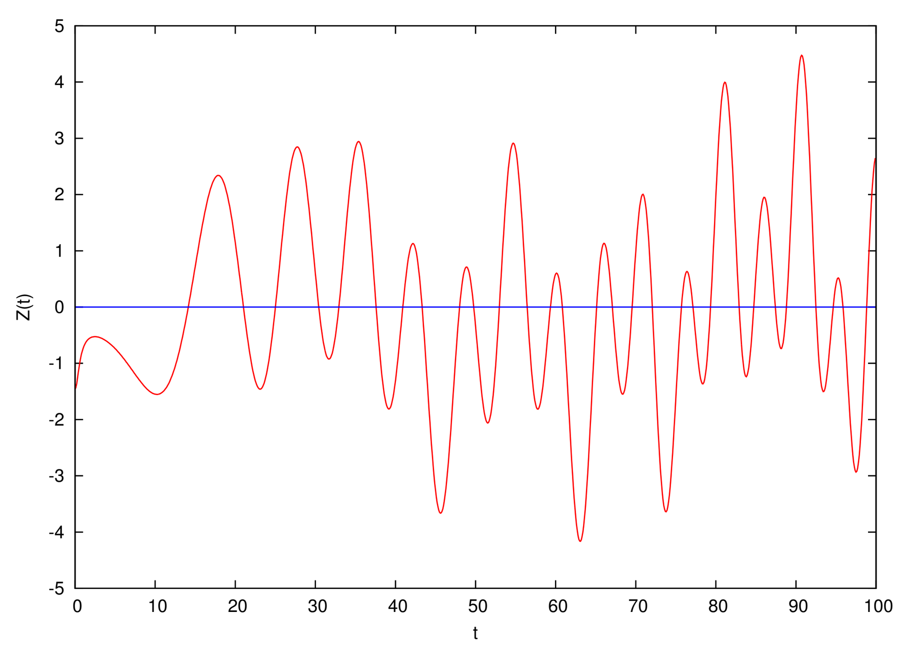


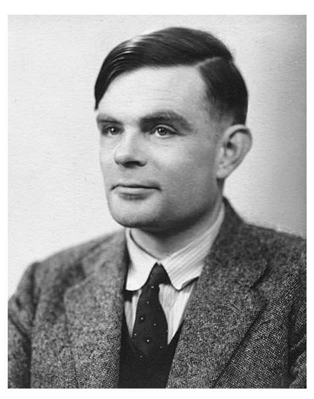
Progress

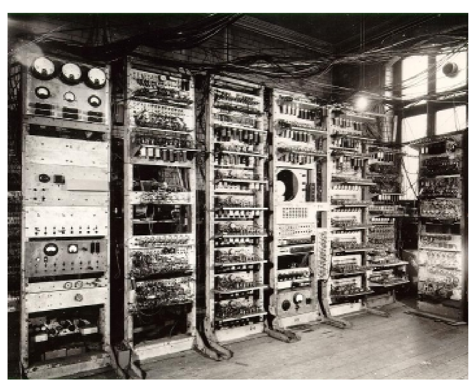
By 2002, Ming-Chit and Tian-Ze had managed to show that the weak Goldbach conjecture held for every odd number $> N_0 = \exp(3100)$ [4]. This is a huge number, way beyond anything we could hope to test, even by computer. On a slightly different tack, Deshouillers et al had managed to show that the conjecture was true if you assumed that the Generalised Riemann Hypothesis (the GRH) was true [1]. Unfortunately, the GRH is even harder to prove (and worth \$1,000,000 if you can do it).

The GRH

The GRH is another mathematical guess, this time about the zeros of the infinite family of Dirichlet L-functions. Although these functions inhabit the whole complex plane, the interesting stuff happens on the "half-line" where $\Re s = 1/2$. The graph below shows how the simplest such function, Riemann's zeta function, behaves on the portion of the half-line from 0 to 100. We can clearly see 29 zeros as it wiggles through the x-axis. The GRH claims that all the zeros of all the Dirichlet L-functions live somewhere on this half-line.







Turing's Method

Using one of the earliest digital computers, the Manchester Mark I, Alan Turing checked the GRH for zeta for a small piece of the half line [6]. To do this, he came up with a way of computing how many zeros there <u>should</u> be on any portion of the half line if the GRH is true.

Turing's method tells us there should be exactly 29 zeros of zeta below height 100. Our graph accounts for them all and thus proves that the GRH holds for zeta to this height.

The Circle Method

The progress towards weak Goldbach we have outlined relied on expressing the problem as an integral and then applying an analytical technique known as the circle method to estimate that integral. Assuming the GRH makes it much easier to get a good estimate, as does only considering numbers larger than some starting point N_0.

In 2013, Harald Helfgott made two significant advances [2]. First, he managed to get the N_0 down to something manageable, a number with a mere 30 digits. Second he showed we didn't need all of the GRH. A finite (but large) subset would suffice.

The First Computation

First we needed to show that Goldbach was right for every odd number between 7 and $N_0 = 10^{30}$. Checking each one individually would take longer than the universe has left. Fortunately, thanks to another huge computational effort [7], we already knew that every even number between 4 and 4×10^{18} was the sum of two primes. So given an odd prime p, we know that all the odd numbers between $p + 4$ and $p + 4 \times 10^{18}$ can be written as the sum of 3 primes.

Now we just needed a "ladder" of primes up to 10^{30} with the rungs less than 4×10^{18} apart, a much more reasonable proposition. Establishing such a ladder took about 40,000 hours on computers at Warwick University and the Paris Observatory [3].

The Second Computation

This left the partial verification of the GRH. Helfgott needed us to check the simplest 29,565,923,838 Dirichlet L-functions. This meant computing each one at many places on the half-line and counting all the zeros up to a certain height. Applying Turing's method, we knew there to be nearly 4×10^{13} zeros and we had to find every one of them on the half-line for Helgott's proof to go through.

To be able to tackle this size of problem, we needed new algorithms [5]. To get the necessary performance, we exploited the miracle that is the Fast Fourier Transform and on average we were able to isolate $> 25,000$ zeros per second on a desktop computer. Even at this rate, we needed 400,000 hours (45 years) of CPU time but by using resources in Bristol (see below) and Lyons, the entire computation was compressed into slightly more than 6 months elapsed.

Where Next?

In the same 1742 letter, Goldbach suggested that every even number > 2 could be written as the sum of two primes. This is known as his "strong" conjecture because, if true, it would automatically imply the weak conjecture we have been considering.

Despite knowing the strong conjecture holds to 4×10^{18}, we still seem to be a long way off proving it holds forever. Not even assuming the GRH does us any good.

References

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[2] Harald A. Helfgott. The ternary Goldbach conjecture is true. `http://arxiv.org/abs/1312.7748`

[3] Harald A. Helfgott and David J. Platt. Numerical verification of the ternary Goldbach conjecture up to $8.875 \cdot 10^{30}$. *Exp. Math.*, 4(22):406–409, 2013.

[4] Liu Ming-Chit and Wang Tian-Ze. On the Vinogradov bound in the three primes Goldbach conjecture. *Acta Arith.*, 105(2002) 133–175.

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[6] Alan M. Turing. Some calculations of the Riemann zeta-function. *Proc. Lond. Math. Soc.*, 3(3):99–117, 1953.

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